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Colored solutions of Yang-Baxter equation from representations of $U_q gl(2)$

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Abstract. We study the Hopf algebra structure and the highest weight representation of a multiparameter version of $U_q gl(2)$. The commutation relations as well as other Hopf algebra maps are explicitly given. We show that the multiparameter universal $\mathcal{R}$ matrix can be constructed directly as a quantum double intertwiner, without using Reshetikhin's transformation. An interesting feature automatically appears in the representation theory: it can be divided into two types, one for generic q, the other for q being a root of unity. When applying the representation theory to the multiparameter universal $\mathcal{R}$ matrix, the so called standard and nonstandard colored solutions $R(\mu, \nu; \mu', \nu')$ of the Yang-Baxter equation is obtained.

1 Introduction

As is well known, the Yang-Baxter equation (YBE) [1, 8] plays an essential role in the study of quantum groups (QG) and quantum algebras (QA) [2, 3, 4, 5, 6, 7, 8], integrable models [9, 10, 11, 12], as well as in the construction of knot or link invariants [13, 14, 15, 16, 17, 18, 19]. For instance, in the Faddeev–Reshetikhin–Takhtajan (FRT) approach [4, 5, 6] to construct quantum groups or quantum algebras, one has to find an R matrix, which is a matrix solution of YBE [8], then using this R matrix as the input, substituting it into the RTT or RLL relations to get the quantum group or quantum algebra as the output.

There are various methods to find the appropriate R matrix. One way is to borrow an $R(u)$ matrix from the integrable model [8] and then taking appropriate limit to remove the spectral parameter u. The second method is to solve the matrix YBE directly [18, 20, 21]. In this approach one usually assumes an R with prescribed nonzero elements, and impose some restrictions on them to find a class of solutions. Some R

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matrices obtained in this way have unexpected interesting features [15, 19, 26, 32], so a number of authors call them "nonstandard" solutions [22, 23, 24].

Many known quantum algebras belong to the category of quasitriangular Hopf algebras (QTHA) [7, 8]. This observation provides us an alternative approach to find the R matrix[38, 40]. When applying representation theory to the universal $\mathcal{R}$ matrix[4] [7, 8] of a QTHA, the desired R matrix is obtained. To get more interesting solutions, people also try different methods to add parameters that appearing in the R matrix . This cause the development of multiparameter deformations [33, 34, 35, 36, 37, 38, 39, 40] Hopf algebras and q-boson realizations [27, 28, 29, 30, 31] with q being a root of unity. These solutions are sometimes called "colored" solutions [19, 32, 38]. Although the q-boson realization method is very powerful in constructing representations of quantum groups or quantum algebras, it's hardly to manifest the Hopf algebra structures.

In this paper we study $U_q gl(2)$. We show that due to the commuting element J, it is possible to introduce additional parameters t, u and v, and hence gives us a multiparameter version of Hopf maps and multiparameter universal $\mathcal{R}$ matrix. We then explain how to get the same $\mathcal{R}$ from quantum double constructions. In this way the Hopf algebra structure is preserved and emphasized. For the representations of $U_q gl(2)$, we only consider the highest weight representations. Under the finite dimension restriction, two categories of representation appears automatically. When applying this representation theory to $\mathcal{R}$, the standard and nonstandard colored solutions are obtained and are consistent with literature's results.

This paper is organized as follows: In section 2, we review some basic definitions and properties of Hopf algebras, quasitriangular Hopf algebras and quantum double. In section 3, different selections of universal $\mathcal{R}$ matrices are given, and compared to the result obtained from Reshetikhin's transformation [33]. In section 4, the highest weight representations are studied and applied to $\mathcal{R}$ to obtain matrix solutions R. In section 5, colored solutions are obtained and compared to the literature's results. Section 6 is devoted to concluding remarks.

2 Hopf algebras, quasi-triangular Hopf algebras and Quantum double

In this section we give brief review of some definitions and properties of Hopf algebras (HA) and quasi-triangular Hopf algebras (QTHA), as well as their relations to the notion of quantum double (QD) [7, 8]. These ideas will then be used in our latter discussions of the multiparameter $U_q gl(2)$.

A. Hopf algebras

A Hopf algebra is an associative algebra A with five basic maps(in this paper, we call them Hopf maps), they are four homomorphisms: $m : A \otimes A \rightarrow A$ (*multiplication*), $\Delta : A \rightarrow A \otimes A$ (*coproduct*), $\eta : C \rightarrow A$ (*inclusion*), $\varepsilon : A \rightarrow C$ (*counit*) and one antihomomorphism: $S : A \rightarrow A$ (*antipode*). They satisfy the following relations for any $a \in A$:

$$\begin{aligned}
&(\Delta \otimes id)\Delta(a) = (id \otimes \Delta)\Delta(a) \\
&(\varepsilon \otimes id)\Delta(a) = (id \otimes \varepsilon)\Delta(a) = id(a) = a \qquad (1) \\
&m(S \otimes id)\Delta(a) = m(id \otimes S)\Delta(a) = \eta \circ \varepsilon(a) = \varepsilon(a)1
\end{aligned}$$

[4] We denote the universal algebraic solution of YBE by $\mathcal{R}$ and the matrix solution by R

where *id* represents the *identity map*. To be more precise, we use the notation $(A, m, \Delta, \eta, \varepsilon, S)$ instead of A to denote a Hopf algebra. With these ideas in mind, the following propsition will be apparent:

Proposition 2.1 *Replacing Δ by $\bar{\Delta} = \Delta^{'}$ and S by $\bar{S} = S^{-1}$, the algebra $(A, m, \bar{\Delta}, \eta, \varepsilon, \bar{S})$ is also a Hopf algebra.*

Here $\Delta^{'}$ denotes the opposite coproduct, which maps any $a \in A$ to $A \otimes A$ as:

$$\Delta^{'}(a) = \sigma \circ \Delta(a) = \sum_i c_i \otimes b_i \ \text{ if } \ \Delta(a) = \sum_i b_i \otimes c_i \tag{2}$$

and S^{-1} is defined as the inverse of S:

$$S(S^{-1}(a)) = S^{-1}(S(a)) = a \tag{3}$$

B. Quasitriangular Hopf algebras

A Quasitriangular Hopf algebra (QTHA)is a Hopf algrbra equipped with an element $\mathcal{R} \in A \otimes A$ which is the solution of the algebraic version of YBE. We start with the definition.

Definition 2.1 *Let $\mathcal{A}$=$(A, m, \Delta, \eta, \epsilon, S)$ be a Hopf algebra and $\mathcal{R}$ (intertwiner) an invertible element in $A \otimes A$, then the pair$(\mathcal{A}, \mathcal{R})$ is called a QTHA if for any $a \in A$ we have*

(i) $\mathcal{R}\Delta(a) = \Delta^{'}(a)\mathcal{R}$

(ii) $(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$

(iii) $(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$

In addition, three further relations are satisfied:

$$\begin{aligned} &\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \\ &(S \otimes id)\mathcal{R} = (id \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1}, \\ &(\varepsilon \otimes id)\mathcal{R} = (id \otimes \varepsilon)\mathcal{R} = 1, \end{aligned} \tag{4}$$

The first line is the Yang-Baxter equation.

As in the case of Hopf algebras, we denote $(A, \mathcal{R}, m, \Delta, \eta, \varepsilon, S)$ as a QTHA. From (i) of definition **2.1**, we immediately find

$$\begin{aligned} &\mathcal{R}\Delta(a) = \Delta^{'}(a)\mathcal{R}, \qquad &(\sigma \circ \mathcal{R})\Delta^{'}(a) = \Delta(a)(\sigma \circ \mathcal{R}), \\ &\mathcal{R}^{-1}\Delta^{'}(a) = \Delta(a)\mathcal{R}^{-1}, \qquad &(\sigma \circ \mathcal{R}^{-1})\Delta(a) = \Delta^{'}(a)(\sigma \circ \mathcal{R}^{-1}). \end{aligned}$$

Define $\mathcal{R}^{(+)} = \sigma \circ \mathcal{R}$, $\mathcal{R}^{(-)} = \mathcal{R}^{-1}$ and $\bar{\mathcal{R}} = \sigma \circ \mathcal{R}^{-1}$, and denote $\Delta^{'}$ as $\bar{\Delta}$, then

$$\begin{aligned} &\mathcal{R}\Delta = \Delta^{'}\mathcal{R}, \qquad &\bar{\mathcal{R}}\Delta = \Delta^{'}\bar{\mathcal{R}}, \\ &\mathcal{R}^{(+)}\bar{\Delta} = \bar{\Delta}^{'}\mathcal{R}^{(+)}, \qquad &\mathcal{R}^{(-)}\bar{\Delta} = \bar{\Delta}^{'}\mathcal{R}^{(-)}. \end{aligned} \tag{5}$$

These observations lead to the following result:

Proposition 2.2 *If* $(A, \mathcal{R}, m, \Delta, \varepsilon, S, \eta)$ *is a QTHA, then* $(A, \bar{\mathcal{R}}, m, \Delta, \varepsilon, S, \eta)$*,* $(A, \mathcal{R}^{(+)}, m, \Delta^{'}, \varepsilon, S^{-1}, \eta)$ *and* $(A, \mathcal{R}^{(-)}, m, \Delta^{'}, \varepsilon, S^{-1}, \eta)$ *are all QTHAs.*

It can be easily proved by using the definition **2.1** and equation (1). This theorem tells us that for a pair $(\Delta\,,\, S)$, there are two universal $\mathcal{R}$ matrices: $\mathcal{R}$ and $\bar{\mathcal{R}} = \sigma \circ \mathcal{R}^{-1}$, both can be used as intertwiner in a QTHA. Now let's turn to the discussion of *quantum double*[7, 8] .

C. Quantum double

Suppose we have a Hopf algebra A, which spanned by basis $\{e_i\}$. By introducing a nondegenerate bilinear form $\langle\ ,\ \rangle$, we can define A's dual algebra A^o, which spanned by $\{e^i\}$; here $\langle e^i\,,\, e_j\rangle = \delta^i_j$. Then all the Hopf maps of A^o can be defined in terms of $\langle\ ,\ \rangle$. Introducing the *intertwiner*:

$$\mathcal{R} = \sum_i e_i \otimes e^i, \tag{6}$$

then the commutation relations between A and A^o can be established via the relation

$$\mathcal{R}\Delta(a) = \Delta^{'}(a)\mathcal{R}, \quad \text{for } \ a \in A \quad \text{or} \quad A^o,$$

which tells us how to expand an $e^i e_j$ type product as the sum of $e_i e^j$ type products. Choosing $\{e_i e^j\}$ as basis, one can "combine" A and A^o to form an enlarged algebra $D(A)$–the *quantum double* of A, and treat A or A^o as its subalgebra. Then $D(A)$ can be proved to be a QTHA equipped with $\mathcal{R} = \sum_i e_i \otimes e^i$ as its intertwiner(universal $\mathcal{R}$ matrix). In other words, a QTHA is a quantum double of its subalgebra. In the next section, we will show that the $U_q gl(2)$ is indeed a quantum double as well as a QTHA.

3 Universal $\mathcal{R}$ matrix of $U_q gl(2)$

We define Our version of $U_q gl(2)$ algebra as a multiparameter QTHA generated by (H, J, X^+, X^-) with the commutation relations

$$\begin{aligned} &[J\,,\, H] = [J\,,\, X^{\pm}] = 0, \\ &[H\,,\, X^{\pm}] = \pm 2X^{\pm}, \\ &[X^+,\, X^-] = \frac{q^H t^{-J} - q^{-H} t^J}{q - q^{-1}}, \end{aligned} \tag{7}$$

and additional Hopf maps:

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta(J) = J \otimes 1 + 1 \otimes J,$$

$$\text{coproduct:} \qquad \Delta(X^+) = q^{-\frac{1}{2}H}(utv)^{\frac{1}{2}J} \otimes X^+ + X^+ \otimes q^{\frac{1}{2}H}(utv^{-1})^{-\frac{1}{2}J}, \tag{8}$$

$$\Delta(X^-) = q^{-\frac{1}{2}H}(u^{-1}tv^{-1})^{\frac{1}{2}J} \otimes X^- + X^- \otimes q^{\frac{1}{2}H}(u^{-1}tv)^{-\frac{1}{2}J},$$

$$\text{antipode:} \qquad S(H) = -H, \;\; S(J) = -J, \;\; S(X^{\pm}) = -q^{\pm 1} v^{\mp J} X^{\pm}, \tag{9}$$

$$\text{counit:} \qquad \varepsilon(H) = \varepsilon(J) = \varepsilon(X^{\pm}) = 0. \tag{10}$$

The universal $\mathcal{R}$ matrix is defined by

$$\mathcal{R} = \mathcal{R}_0 \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{\{n\}_{q^2}!} q^{n(n-1)} q^{\frac{n}{2}(H\otimes 1 - 1 \otimes H)} ((uvt^{-1})^{\frac{1}{2}J} X^-)^n \otimes ((uv^{-1}t)^{\frac{1}{2}J} X^+)^n, \tag{11}$$

where

$$\mathcal{R}_0 = q^{-\frac{1}{2}H\otimes H} t^{\frac{1}{2}(H\otimes J + J\otimes H)} u^{\frac{1}{2}(H\otimes J - J\otimes H)}, \tag{12}$$

t, u and v are arbitrary parameters and $\{n\}_{q^2}$, $\{n\}_{q^2}!$ are defined by

$$\begin{aligned} \{n\}_{q^2} &= \frac{1-q^{2n}}{1-q^2} = q^{n-1}[n]_q, \\ \{n\}_{q^2}! &= \prod_{j=1}^{n} \{j\}_{q^2} = q^{\frac{1}{2}n(n-1)}[n]_q!, \end{aligned} \tag{13}$$

with $\{0\}_{q^2}! = [0]_q! = 1$. Note that the commuting element J appearing in this algebra causes the expression of Hopf maps has many different choices. For example, the parameter t in the last commutation relation of (7) is not essential. One can always absorb the factor t^{-J} into q^H by defining $q^H t^{-J} = q^{H'}$ and rename H' by H. However, in order to reflect the fact that J can be arbitrarily 'mixed' with H, in this paper we shall always retain the parameter t. On another hand, two arbitrary parameters u and v are allowed to appearing in the definitions of Δ and S, although they do not explicitly appear in (7).

Note that under the transformation:

$$\tilde{X^{\pm}} = v^{\mp\frac{1}{2}J} X^{\pm}, \tag{14}$$

the commutation relations (7) will not change its form. Moreover, 14 simplifies the form of Δ and S on $X^{\pm}$:

$$\Delta(\tilde{X^{\pm}}) = (q^{-\frac{1}{2}H} t^{\frac{1}{2}J}) u^{\pm\frac{1}{2}J} \otimes \tilde{X^{\pm}} + \tilde{X^{\pm}} \otimes (q^{\frac{1}{2}H} t^{-\frac{1}{2}J}) u^{\mp\frac{1}{2}J}, \tag{15}$$

$$S(\tilde{X^{\pm}}) = -q^{\frac{1}{2}H} \tilde{X^{\pm}} q^{-\frac{1}{2}H} = -q\tilde{X^{\pm}}. \tag{16}$$

Furthermore, the universal $\mathcal{R}$ matrix now becomes:

$$\mathcal{R} = \mathcal{R}_0 \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[n]_q!} q^{-\frac{1}{2}n(n-1)} ((ut^{-1})^{\frac{1}{2}J} q^{\frac{1}{2}H} \tilde{X^{-}})^n \otimes ((ut)^{\frac{1}{2}J} q^{-\frac{1}{2}H} \tilde{X^{+}})^n. \tag{17}$$

In the following, we shall use $\tilde{X^{\pm}}$ as generators.

As stated in the last section,corresponding to the same pair (Δ , S), there is another universal $\mathcal{R}$ matrix:

$$\bar{\mathcal{R}} = \bar{\mathcal{R}}_0 \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{1}{2}n(n-1)} ((ut)^{-\frac{1}{2}J} q^{\frac{1}{2}H} \tilde{X^{+}})^n \otimes ((u^{-1}t)^{\frac{1}{2}J} q^{-\frac{1}{2}H} \tilde{X^{-}})^n \tag{18}$$

with

$$\bar{\mathcal{R}}_0 = \sigma \circ \mathcal{R}_0^{-1} = q^{\frac{1}{2}H\otimes H} t^{-\frac{1}{2}(H\otimes J + J\otimes H)} u^{\frac{1}{2}(H\otimes J - J\otimes H)}. \tag{19}$$

Similarly, if we use $\bar{\Delta} = \Delta'$ and $\bar{S} = S^{-1}$ as *coproduct* and *antipode* respectively then, for the pair ($\bar{\Delta}$, $\bar{S}$), we have the following two universal $\mathcal{R}$ matrices:

$$\begin{aligned} \mathcal{R}^{(+)} &= \mathcal{R}_0^{(+)} \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[n]_q!} q^{-\frac{1}{2}n(n-1)} ((ut)^{\frac{1}{2}J} q^{-\frac{1}{2}H} \tilde{X^{+}})^n \otimes ((ut^{-1})^{\frac{1}{2}J} q^{\frac{1}{2}H} \tilde{X^{-}})^n, \\ \mathcal{R}_0^{(+)} &= \sigma \circ \mathcal{R}_0 = q^{-\frac{1}{2}H\otimes H} t^{\frac{1}{2}(H\otimes J + J\otimes H)} u^{-\frac{1}{2}(H\otimes J - J\otimes H)}, \end{aligned} \tag{20}$$

and

$$\mathcal{R}^{(-)} = \mathcal{R}_0^{(-)} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{1}{2}n(n-1)} ((u^{-1}t)^{\frac{1}{2}J}) q^{-\frac{1}{2}H} \tilde{X^-})^n \otimes ((ut)^{-\frac{1}{2}J} q^{\frac{1}{2}H} \tilde{X^+})^n,$$
$$\mathcal{R}_0^{(-)} = \mathcal{R}_0^{-1} = q^{\frac{1}{2}H\otimes H} t^{-\frac{1}{2}(H\otimes J + J\otimes H)} u^{-\frac{1}{2}(H\otimes J - J\otimes H)}. \tag{21}$$

These universal $\mathcal{R}$ matrices can be compared to the literature [37]-[40]. However, since different authors adopt different conventions in the definition of Δ and S, thus we have to properly choose one $\mathcal{R}$ from the set $\{\mathcal{R}, \bar{\mathcal{R}}, \mathcal{R}^{(+)}, \mathcal{R}^{(-)}\}$. Note that the parameters t, u and v can be freely choosen due to the fact that there exist the commuting generator J. If we use $H_1 = H - \alpha_1 J$ and $H_2 = H - \alpha_2 J$ as generators instead of H and J where $q^{\alpha_1} = u^{-1}t$ and $q^{\alpha_2} = ut$, then the universal $\mathcal{R}$ matrix can be expressed as the following simple form (here we drop a trivial commuting factor $q^{-\frac{1}{2}\alpha_1\alpha_2 J\otimes J}$):

$$\mathcal{R} = q^{-\frac{1}{2}H_1\otimes H_2} \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[n]_q!} q^{-\frac{1}{2}n(n-1)} (q^{\frac{1}{2}H_1} \tilde{X^-})^n \otimes (q^{-\frac{1}{2}H_2} \tilde{X^+})^n, \tag{22}$$

which is very similar to the universal $\mathcal{R}$ matrix of $U_q sl(2)$:

$$\mathcal{R}_{sl} = q^{-\frac{1}{2}H\otimes H} \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[n]_q!} q^{-\frac{1}{2}n(n-1)} (q^{\frac{1}{2}H} \tilde{X^-})^n \otimes (q^{-\frac{1}{2}H} \tilde{X^+})^n. \tag{23}$$

In fact, the similarity is not an accident but a consequence of QD. To see this, we first replace the generators $\tilde{X^+}$ and $\tilde{X^-}$ by e and f [7]:

$$e = q^{-\frac{H_2}{2}} \tilde{X^+} , \quad f = q^{\frac{H_1}{2}} , \tilde{X^-} \tag{24}$$

and then the equation (7)-(10) become

$$[H_{1,2}\, , \, e] = 2e \; , \quad [H_{1,2}\, , \, f] = -2f \; ,$$
$$[e\, , \, f] = \frac{q^{H_1} - q^{-H_2}}{q^2 - 1} \; , \tag{25}$$
$$\Delta(H_{1,2}) = H_{1,2} \otimes 1 + 1 \otimes H_{1,2} \; , \; \Delta(1) = 1 \otimes 1 \; ,$$
$$\Delta(e) = e \otimes 1 + q^{-H_2} \otimes e \; , \; \Delta(f) = 1 \otimes f + f \otimes q^{H_1}. \tag{26}$$
$$S(H_{1,2}) = -H_{1,2} \; , \; S(e) = -q^{H_2} e \; , \; S(f) = -f q^{-H_1} \; , \; S(1) = 1, \tag{27}$$
$$\varepsilon(H_{1,2}) = \varepsilon(e) = \varepsilon(f) = 0 \; , \; \varepsilon(1) = 1. \tag{28}$$

These equations provide us the coefficients in the construction of quantum double. Now, choosing the lower Borel subalgebra of $U_q gl(2)$

$$U_q\mathcal{B}_- = span\{H_1^n f^m\}_{n,m=0}^{\infty}$$

as the Hopf algebra A in the quantum double construction, then by applying the same method as Tjin did in [7], we will find that A^o can be identified to the upper Borel subalgebra

$$U_q\mathcal{B}_+ = span\{H_2^n e^m\}_{n,m=0}^{\infty}$$

and finally obtain the quantum double $D(A)$ as $U_q gl(2)$.

Note that in the case of $U_q sl(2)$, the dual element of H can only be identified to an element proportional to itself. However, in the $U_q gl(2)$ case, the existence of commuting element J makes it possible to identify the dual element of H_1 as H_2, with

$$H_1 - H_2 \propto J,$$

thus establish the universal $\mathcal{R}$ matrix in equation (22).

The same multiparameter universal $\mathcal{R}$ matrix can also be obtained in a different way. Denote $\mathcal{R}$ in (17) as $\mathcal{R}(H_1\,,\,H_2)$. Let $u=1$ and define $q^{H'} = q^H t^{-J}$, we obtain a single-parameter $U_q gl(2)$ and universal $\mathcal{R}$ matrix denoted by $\mathcal{R}(H'\,,\,H')$:

$$\mathcal{R}(H'\,,\,H') = q^{-\frac{1}{2}H'\otimes H'} \sum_{n=0}^{\infty} \frac{(1-q^2)^n}{[n]_q!} q^{-\frac{1}{2}n(n-1)} (q^{\frac{1}{2}H'}\tilde{X^-})^n \otimes (q^{-\frac{1}{2}H'}\tilde{X^+})^n. \tag{29}$$

According to the procedure introduced by Reshetikhin [33]: if we can find an element $F = \sum_i f^i \otimes f_i \in U_q gl(2) \otimes U_q gl(2)$ satistying

$$\begin{aligned}(\Delta\otimes id)F = F_{13}F_{23}, &\qquad (id\otimes\Delta)F = F_{13}F_{12},\\ F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}, &\qquad F_{12}F_{21} = 1,\end{aligned} \tag{30}$$

then we can build a multiparameter version of this QTHA and thus obtain a multiparameter universal $\mathcal{R}$ matrix:

$$\mathcal{R}^{(F)} = F^{-1}\mathcal{R}(H',H')F^{-1}. \tag{31}$$

One can check that

$$F = u^{-\frac{1}{4}(H\otimes J - J\otimes H)} \tag{32}$$

can be used to do this construction and the Hopf maps defined in (7)-(10) and the universal $\mathcal{R}$ matrix in (17) will be recovered. Note that in the expression of $\mathcal{R}_0$ (cf. equation (12)), the exponent of the parameter u has an *antisymmetric* form, which can be obtained from Reshetikhin's transformation. On the other hand, the exponent of the parameter t has a symmetric form which comes from the third formula of (7), and *cannot* be obtained from Reshetikhin's transformation.

4 The highest weight representations of $U_q gl(2)$

For the representation theory, we only study the highest weight representations [15, 30, 38]. Let π be the map from $U_q gl(2)$ to a $m-$dimensional ($m \geq 2$) representation:

$$\begin{aligned}&\pi(J) = \lambda\mathbf{1}, \quad \pi(H) = \mu\mathbf{1} + \sum_{i=1}^{m}(m-2i+1)\,e_{ii},\\ &\pi(\tilde{X^+}) = \sum_{i=1}^{m-1} a_i\,e_{i,i+1}, \quad \pi(\tilde{X^-}) = \sum_{i=1}^{m-1} b_i\,e_{i+1,i},\end{aligned} \tag{33}$$

here e_{ij} represents the matrix basis ($(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$) and $\mathbf{1}$ denots the unit matrix. Our strategy is to find a proper choice of parameters λ, μ, $\{a_i, b_i\}_{i=0}^{m}$ such that they will give us the highest weight representations of $U_q gl(2)$. Substituting these expressions to (7), we get

$$a_i b_i = [i]_q \left(\frac{q^\mu q^{m-i} t^{-\lambda} - q^{-\mu} q^{i-m} t^{\lambda}}{q-q^{-1}}\right), \quad i = 1, 2, \ldots, m-1, \tag{34}$$

here b_i does not have any prior relation to a_i. Equation (34) naturally comes from the commutation relation (7) of $U_q gl(2)$. Let $t^\lambda = q^\tau$, (34) can now be rewritten as:

$$a_i b_i = [i]_q[\mu - \tau + m - i]_q, \quad i = 1, 2, \ldots, m-1.$$

For $i = m-1$, comparing with another expression (also obtained from (7)):

$$a_{m-1}b_{m-1} = -[\mu - \tau + 1 - m]_q,$$

and using the identities:

$$[x]_q^2 - [y]_q^2 = [x-y]_q[x+y]_q,$$
$$[x]_q[y]_q = [\frac{x+y}{2}]_q^2 - [\frac{x-y}{2}]_q^2,$$

we find

$$[\mu - \tau]_q[m]_q = 0. \tag{35}$$

This result thus gives us two kinds of the highest weight representation:

- **Type a.** If $q^{2(\mu-\tau)} = 1$ or $q^{2\mu}t^{-2\lambda} = 1$, then q can be any complex number.
- **Type b.** If μ, τ or $q^{2\mu}t^{-2\lambda}$ are arbitrary complex numbers, then we must have the restriction $[m]_q = 0$ or $q^{2m} = 1$. In other words, q must be restricted to roots of unity.

Now let's consider two simple examples. First the $m = 2$ case:

$$\pi(H) = \begin{pmatrix} \mu+1 & 0 \\ 0 & \mu-1 \end{pmatrix}, \quad \pi(J) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\pi(\tilde{X^+}) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad \pi(\tilde{X^-}) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \tag{36}$$

$$ab = \frac{q^{\mu+1}t^{-\lambda} - q^{-\mu-1}t^\lambda}{q - q^{-1}}. \tag{37}$$

The 4×4 matrix solutions R of YBE can be obtained via the representation $R = (\pi \otimes \pi)\mathcal{R}$:

$$R = q^{-\frac{1}{2}(\mu^2-1)}t^{\lambda\mu} \begin{pmatrix} q^{-1}(q^{-\mu}t^\lambda) & 0 & 0 & 0 \\ 0 & u^\lambda & 0 & 0 \\ 0 & (q^{-1}-q)ab & u^{-\lambda} & 0 \\ 0 & 0 & 0 & q^{-1}(q^\mu t^{-\lambda}) \end{pmatrix}. \tag{38}$$

Let $q^\mu t^{-\lambda} = q^{-1}s$, $u^\lambda = \gamma$ and drop the factor $q^{-\frac{1}{2}(\mu^2-1)}t^{\lambda\mu}$, we have

$$R = \begin{pmatrix} s^{-1} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & s^{-1}-s & \gamma^{-1} & 0 \\ 0 & 0 & 0 & q^{-2}s \end{pmatrix}. \tag{39}$$

According to previous discussion, this R matrix in fact represents two solutions, which are

$$R_{(1)} = \begin{pmatrix} s^{-1} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & s^{-1}-s & \gamma^{-1} & 0 \\ 0 & 0 & 0 & s^{-1} \end{pmatrix}, \quad R_{(2)} = \begin{pmatrix} s^{-1} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & s^{-1}-s & \gamma^{-1} & 0 \\ 0 & 0 & 0 & -s \end{pmatrix}. \tag{40}$$

When q is generic, we must have $q^{-2}s^2 = 1$ which gives us solution $R_{(1)}$. On the other hand, if s is arbitrary, we have $q^4 = 1$ which implies $q^2 = -1$ ($q^2 = 1$ is ruled out since that will cause $ab \to \infty$)and gives us solution $R_{(2)}$. Next, we consider the $m = 3$ case,

$$\pi(H) = \begin{pmatrix} \mu+2 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu-2 \end{pmatrix}, \quad \pi(J) = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\pi(\tilde{X^+}) = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi(\tilde{X^-}) = \begin{pmatrix} 0 & 0 & 0 \\ b_1 & 0 & 0 \\ 0 & b_2 & 0 \end{pmatrix}, \tag{41}$$

$$a_1 b_1 = [\mu - \tau + 2]_q = \left(\frac{q^{\mu+2}t^{-\lambda} - q^{-\mu-2}t^{\lambda}}{q - q^{-1}} \right), \quad t^\lambda = q^\tau,$$

$$a_2 b_2 = [2]_q[\mu - \tau + 1]_q = (q + q^{-1}) \left(\frac{q^{\mu+1}t^{-\lambda} - q^{-\mu-1}t^{\lambda}}{q - q^{-1}} \right). \tag{42}$$

Let $q^{-\mu}t^{\lambda} = q^2 s^{-2}$, $u^\lambda = \gamma$ and remove the factor $q^{-\frac{1}{2}\mu^2}t^{\lambda\mu}$, we get

$$R = \begin{pmatrix} A_1 & 0 & 0 \\ B_1 & A_2 & 0 \\ C & B_2 & A_3 \end{pmatrix}, \tag{43}$$

where A_1, A_2, A_3, B_1, B_2 and C are 3×3 matrices:

$$A_1 = \begin{pmatrix} q^2 s^{-4} & 0 & 0 \\ 0 & q^2 s^{-2}\gamma & 0 \\ 0 & 0 & q^2\gamma^2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} q^2 s^{-2}\gamma^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2}s^2\gamma \end{pmatrix},$$

$$A_3 = \begin{pmatrix} q^2\gamma^{-2} & 0 & 0 \\ 0 & q^{-2}s^2\gamma^{-1} & 0 \\ 0 & 0 & q^{-6}s^4 \end{pmatrix}, \tag{44}$$

$$B_1 = \begin{pmatrix} 0 & q^2(s^{-4}-1) & 0 \\ 0 & 0 & (1-q^2)\gamma a_2 b_1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & (1-q^2)\gamma^{-1}a_1 b_2 & 0 \\ 0 & 0 & (1+q^{-2})(1-q^{-2}s^4) \\ 0 & 0 & 0 \end{pmatrix}, \tag{45}$$

$$C = \begin{pmatrix} 0 & 0 & (s^{-4}-1)(q^2-s^4) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{46}$$

This result also gives us two kinds of R matrices. When $(q/s)^4 = 1$, we have the type a. solution (the standard solution), whereas in the situation $(q/s)^4 \neq 1$, we have $q^6 = 1 \to 1 + q^2 + q^4 = 0$ which gives us type b. solution (the nonstandard solution)

respectively. Notice that the factors a_1b_2 and a_2b_1 appearing in B_1 and B_2 cannot be uniquely determined in terms of q, γ, s only, whereas their product $(a_1b_2a_2b_1) = (a_1b_1a_2b_2)$ is unique. For a general integer m, after removing the factor $q^{-\frac{1}{2}\mu^2}t^{\lambda\mu}$, and let

$$q^{\mu}t^{-\lambda} = (q^{-1}s)^{m-1}\,, \qquad u^{\lambda} = \gamma, \tag{47}$$

we have

$$R = q^{\frac{1}{2}(m-1)^2}s^{-(m^2-1)}\sum_{n=0}^{m-1}\frac{(1-q^2)^n}{\{n\}_{q^2}!}q^n\sum_{i,j=1}^{m-n}q^{-2(i-1)(j-1)-n(i+j)}$$

$$s^{(m-1)(i+j+n)}\gamma^{-(i-j)}(a_jb_i)\cdots(a_{j+n-1}b_{i+n-1})\,e_{i+n,i}\otimes e_{j,j+n}, \tag{48}$$

where

$$\begin{aligned} a_ib_i &= [i]_q\left(\frac{q^{\mu}t^{-\lambda}q^{m-i} - q^{-\mu}t^{\lambda}q^{i-m}}{q-q^{-1}}\right) \\ &= [i]_q\left(\frac{s^{m-1}q^{1-i} - s^{1-m}q^{i-1}}{q-q^{-1}}\right) \end{aligned} \tag{49}$$

and the identity

$$(q^{2\mu}t^{-2\lambda}-1)[m]_q = \left((\frac{s}{q})^{2(m-1)}-1\right)[m]_q = 0 \tag{50}$$

is hold. Here we define: $(a_jb_i)\cdots(a_{j+n-1}b_{i+n-1}) \equiv 1$ when $n=0$.

5 Colored solutions of Yang-Baxter equation

In order to obtain a colored solution of YBE via representation, we have to prepare two representations of $U_qgl(2)$ [30, 38]: $\pi_1 = \pi^{\mu,\lambda}$ and $\pi_2 = \pi^{\mu',\lambda'}$ acting on the former and later entries associated with tensor product $\otimes$ respectively. Then the colored solution is given by

$$R(\mu,\lambda;\mu',\lambda') = (\pi_1\otimes\pi_2)\mathcal{R} \tag{51}$$

Now let's calculate $R(\mu,\lambda;\mu^{'},\lambda^{'})$. For the former entry associated with $\otimes$, we have

$$\pi_1(H) = \sum_{i=1}^{m}(\mu+m-2i+1)\,e_{ii}, \qquad \pi_1(J) = \lambda\mathbf{1} = \lambda\sum_{i=1}^{m}e_{ii},$$

$$\pi_1(\tilde{X^-}) = \sum_{i=1}^{m-1}b_i\,e_{i+1,i},$$

and for the later entry, we have

$$\pi_2(H) = \sum_{i=1}^{m}(\mu'+m-2i+1)\,e_{ii}, \qquad \pi_2(J) = \lambda'\mathbf{1} = \lambda'\sum_{i=1}^{m}e_{ii},$$

$$\pi_2(\tilde{X^+}) = \sum_{i=1}^{m-1}{a'}_i\,e_{i,i+1}.$$

Here,

$$R(\mu,\lambda\,;\,\mu',\lambda') = f(\mu,\lambda\,;\,\mu',\lambda') \sum_{n=0}^{m-1} \frac{(1-q^2)^n}{\{n\}_{q^2}!} q^n (ss')^{\frac{n}{2}(m-1)} (\frac{\gamma}{\gamma'})^{\frac{n}{2}}$$

$$\sum_{i,j=1}^{m-n} q^{-2(i-1)(j-1)-n(i+j)} ((s')^{m-1}(\gamma')^{-1})^i (s^{m-1}\gamma)^j$$

$$(a'_j b_i)\cdots(a'_{j+n-1} b_{i+n-1})\, e_{i+n,i} \otimes e_{j.j+n}, \tag{52}$$

and s , s', γ , γ' are defined by

$$(\frac{s}{q})^{m-1} = q^{\mu} t^{-\lambda}, \quad (\frac{s'}{q})^{m-1} = q^{\mu'} t^{-\lambda'}, \quad \gamma = u^{\lambda}, \quad \gamma' = u^{\lambda'}. \tag{53}$$

and the factor

$$f(\mu,\lambda\,;\,\mu',\lambda') = q^{-\frac{1}{2}\mu\mu'} t^{\frac{1}{2}(\mu\lambda'+\mu'\lambda)} u^{\frac{1}{2}(\mu\lambda'-\mu'\lambda)} q^{\frac{1}{2}(m-1)^2} (ss')^{-\frac{1}{2}(m^2-1)} (\frac{\gamma'}{\gamma})^{\frac{1}{2}(m+1)} \tag{54}$$

is irrelevant and can be dropped.

As discussed in the last section, there are two different types of solution: type a(q is generic) and type b(q is a root of unity). When $m=2$, let's compare our results with Hlavatý's solutions [41] (see also [26]):

$$R_1(\lambda,\mu) = \phi(\lambda,\mu) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^+(\lambda) & 0 & 0 \\ 0 & (1-k)\xi(\lambda)/\xi(\mu) & k/p^+(\mu) & 0 \\ 0 & 0 & 0 & p^+(\lambda)/p^+(\mu) \end{pmatrix}, \tag{55}$$

$$R_2(\lambda,\mu) = \phi(\lambda,\mu) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p^+(\lambda) & 0 & 0 \\ 0 & W(\lambda,\mu) & p^-(\mu) & 0 \\ 0 & 0 & 0 & -p^+(\lambda)p^-(\mu) \end{pmatrix}, \tag{56}$$

where

$$W(\lambda,\mu) = (1-p^+(\lambda)p^-(\lambda))\xi(\lambda)/\xi(\mu) \tag{57}$$

with $\xi(\lambda)$ is an arbitrary function.

1. For type a:

$$R_a = q^2(\gamma/\gamma') \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q\gamma & 0 & 0 \\ 0 & \pm(1-q^2)(\gamma/\gamma')^{\frac{1}{2}} & q/\gamma' & 0 \\ 0 & 0 & 0 & \gamma/\gamma' \end{pmatrix}, \tag{58}$$

which becomes R_1 when we define $p^+(\lambda) = q\gamma$, $p^+(\mu) = q\gamma'$,$k = q^2$, $\xi(\lambda)/\xi(\mu) = \pm(\gamma/\gamma')^{\frac{1}{2}}$.

2. For type b:

$$R_b = (ss')(\frac{\gamma}{\gamma'}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s\gamma & 0 & 0 \\ 0 & -2q(ss')^{\frac{1}{2}}(\gamma/\gamma')^{\frac{1}{2}}a'b & s'/\gamma' & 0 \\ 0 & 0 & 0 & -ss'(\gamma/\gamma') \end{pmatrix}, \tag{59}$$

here $q^2 = -1$, a' , b are arbitrary C numbers. Let $p^+(\lambda) = s\gamma$, $p^+(\mu) = s'\gamma'$, $p^-(\lambda) = s/\gamma$, $p^-(\mu) = s'/\gamma'$, we get the diagonal part of R_2. Furthermore, rewriting $a'b = a'ab/a$, and using the relation $ab = (s - s^{-1})/(q - q^{-1})= (q/2s)(1 - s^2)$ and define

$$\frac{\xi(\lambda)}{\xi(\mu)} = \frac{[(\gamma/s)^{\frac{1}{2}}/a]}{[(\gamma'/s')^{\frac{1}{2}}/a']} \tag{60}$$

we obtain $W(\lambda,\ \mu) = -2q(ss')^{\frac{1}{2}}(\gamma/\gamma')^{\frac{1}{2}}a'b$, which leads to the-non standard solution R_2.

Another interesting application is to compare our solution with that given in [30]. Their universal $\mathcal{R}$ matrix (4.1) is our $\bar{\mathcal{R}}$. The equivalence can be easily understood by the replacements:

$$2\hat{N} - \lambda_1 \longrightarrow H_1, \quad 2\hat{N} - \lambda_2 \longrightarrow H_2, \tag{61}$$

$$a^\dagger \cdot \alpha(\hat{N}) \longrightarrow \tilde{X^+}, \quad a \cdot \beta(\hat{N}) \longrightarrow \tilde{X^-}. \tag{62}$$

The additional relation

$$\alpha_i(\hat{N} - 1) \cdot \beta_i(\hat{N}) = [\lambda_i + 1 - \hat{N}]_q \tag{63}$$

appearing in [30] is a consistency condition, just like our equations (49) and (50). Therefore, without explicit calculation, the solutions obtained in [30] are the same as (52).

When comparing the solution (52) with those in [19, 32, 38, 40], one should be aware of the definitions and conventions between ours and theirs(in particular, some authors define our RP or PR as their R, P represents the permutation matrix). Others even adopt different convention in the definitions of Δ and S. Therefore, one should first properly choose a correct convention of $\{\Delta, S\}$ and definition of $\mathcal{R}$ or R.

6 Concluding remarks

We have studied the Hopf algebra structure and representation theory of a multiparameter version of $U_q gl(2)$. We show that the YBE can be solved directly in the QTHA framework, without introducing additional tricks or doing any transformations. The interesting feature of highest weight representation shows that there exist two kinds of representations. A large class of Borel type solutions R can be obtained via the highest weight representation, including standard and nonstandard colored solutions. However, in this paper we have not yet discussed the cyclic representation [42, 29, 43] of $U_q gl(2)$ for q being a root of unity. We also have not explored what will happen to the $U_q gl(2)$ algebra itself and its universal $\mathcal{R}$ matrix when q is a root of unity [44, 45]. We leave these discussions to another publication.

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